Fair Division: How to Cut a Cake

We are going to consider problems of fair division. Similar problems will arise in many situations, from divorce settlements to peace treaties, but we will start off by looking at a traditional example, in which Alice and Bob (and later some of their friends) have a cake and want to divide it between them as fairly as possible. Any ideas for how they should do this? Bear in mind that different people might like different parts of the cake more than others (Alice might really like icing, for example, whilst Bob might hate it).

Definition 1. *We say that a method of dividing a cake between n people is **fair** if each person can guarantee that they have at least $\frac{1}{n}^{th}$ of the cake (by their own measure).*

Cut and Choose

Alice cuts the cake into two pieces, and Bob chooses whichever piece he likes.

This is an old, old method of division. This quote is from holocaust survivor Primo Levi:

> When the letter was finished, Grigo pulled out a ration of bread and handed it to me together with the knife. It was the custom, indeed the unwritten law, that in all payments based on bread one of the contracting parties must cut the bread and the other choose, because in this way the person who cuts is induced to make the portions as equal as possible. I was surprised that Grigo already knew the rule, but then I thought that perhaps it applied also outside the Camp in the to me unknown world from which Grigo came.

Theorem 2. *Cut and Choose results in a fair division of the cake.*

Proof. Alice can divide the cake exactly in half, thus guaranteeing that she gets a piece which she believes is exactly one half of the cake. Bob has a choice of two pieces. He must believe that at least one of these two pieces is at least half of the cake, so Bob can choose this piece, also guaranteeing that he gets least half of the cake. □

More People

Now Charlie comes along. Charlie also wants some cake. Do you have any ideas for how Alice, Bob and Charlie can share the cake between them?

Exercise 3. *Find a protocol for three people to divide a cake fairly between them. Prove that it is fair.*

> **Example 4.** *Here is a method that isn't fair. Alice divides the cake into three pieces, Bob chooses his piece, then Charlie divides the remainder of the cake into two equal pieces, letting Alice choose one of the two. Can you see why this isn't fair?*

What if Davina, Eric, Frank and Glenda want to join in? Can you extend this method to share a cake between everyone? Can you prove that it is fair?

> **Exercise 5.** *Find a protocol for dividing a cake between n people, show that it is fair.*

If there are 10 people, how many cuts does this method need? Can you think of any methods for dividing a cake between large numbers of people which don't use too many cuts? Can you prove that these methods are fair?

Envy Free Division

> **Definition 6.** *We say that a method of dividing a cake is **envy free** if no-one thinks that anyone else has a bigger piece of cake.*

> **Exercise 7.** *Prove that any envy-free division protocol is also fair. Is the converse true? (i.e. , is any fair division protocol also envy-free?)*

> **Exercise 8.** *Is 'Cut and Choose' envy free? What about the other methods you had for dividing a cake between more people?*

> **Exercise 9.** *Design an envy-free protocol for dividing a cake between 3 people. Is it possible to extend this method to 4 people?*