Solutions

Solution to exercise ??

There are lots of ways of doing this. I will suggest two below. You might have come up with a different one. If you did, remember to check that it is **fair**.

Method 1

Have Alice divide the cake into two pieces, and let Bob choose one of the two. Now, let Bob and Alice each divide their piece into 3 equal sized pieces, and let Charlie choose one piece of each of them.

Proof Method 1 is fair. Alice gets exactly $\frac{2}{3}$ of $\frac{1}{2}$ of the cake, which is $\frac{1}{3}$ of the cake. Bob gets $\frac{2}{3}$ of a piece of cake which he thought was at least $\frac{1}{2}$ of the total, which is at least $\frac{1}{3}$. It's slightly trickier to prove that Charlie gets at least $\frac{1}{3}$. Let's say that he believes that Alice has some fraction p of the cake, so that Bob has $1-p$. Then Charlie gets at least $\frac{1}{3}p + \frac{1}{3}(1-p) = \frac{1}{3}$ of the total. □

Method 2

Alice cuts a piece which she believes to be $\frac{1}{3}$ of the cake. This is then shown to Bob. If he believes it is more than $\frac{1}{3}$, he may reduce it, it is then shown to Charlie. The last person to touch the piece then takes it, and the other two play Cut and Choose for the remainder.

Proof that Method 2 is fair. Whoever touched the piece last can ensure they have at least $\frac{1}{3}$ of the cake by simply making sure that they never reduce the piece so that it is less than $\frac{1}{3}$. The other two players both agree that this piece is at most $\frac{1}{3}$, so will be playing Cut and Choose with at least $\frac{2}{3}$ of the cake, thus guaranteeing themselves at least $\frac{1}{3}$. □

Both of the above methods generalise very easily to more than 3 players. You can prove that they are still fair by *induction.* You may have come up with a more complicated procedure which is harder to generalise. This is fine. If you did manage to come up with a different procedure which you were able to generalise, check that it is **fair**.

Solution to Problem ??

Any envy-free division protocol is also fair. If a division is envy-free then everyone believes that they have at least as much as everyone else. But then they must have at least $\frac{1}{n}$ of the cake (as at least one person must have at least $\frac{1}{n}$ of the cake, so the division is also fair.

Solution to problem ??

Cut and Choose is envy-free. Assume Alice cuts the cake in half, and Bob chooses the larger piece (remember, they might disagree on what constitutes half a cake). Then Alice thinks Bob has exactly half, while Bob thinks Alice has at most half, so neither can be envious.

Problem ??: an envy-free division protocol

This is *very* tricky! Steinhaus, who was the first mathematician to consider the cake cutting problem in a formal setting, was convinced it couldn't be done. There are several methods however. The following works for three people (it's pretty complicated, you might need to draw a diagram of where the different bits of the cake go in order to follow it). It doesn't generalise in any obvious way to 4 people (in fact, no-one knows of a discrete algorithm for 4 people with a bounded number of cuts!)

1. Alice cuts the cake into three pieces.
2. Bob trims the largest piece so that it is the same size as the second largest. If he thinks the largest is the same size as the second largest, the players just choose pieces in the order: Charlie, Bob, Alice.
3. The players choose pieces in the order Charlie, Bob, Alice, with the restriction that if Charlie doesn't choose the trimmed piece, then Bob must. (the trimmings are left to one side for now)
4. Now, one of Bob and Charlie has the trimmed piece. Let's call the one who has it T and the one who doesn't have it NT. NT cuts the trimmings into three equal pieces.
5. The players choose their share of the trimmings in the order T, Alice, NT.

Proof that this algorithm is envy-free. I leave this proof as an exercise! It isn't actually too hard, remember that Alice already believes that her piece is the same size as T's piece plus *all* of the trimmings, so she will never be envious of T. Similar arguments show that no-one can be envious of anyone else. □